CRITICISMS AND DISCUSSIONS.

THE ARITHMETICAL PYRAMID OF MANY DIMENSIONS

AN EXTENSION OF PASCAL'S ARITHMETICAL TRIANGLE TO THREE AND MORE DIMENSIONS, AND ITS APPLICATION TO COMBINATIONS OF MANY VARIATIONS.

I.

In 1665 Pascal wrote his *Traité du triangle arithmétique* and showed that the system of numbers there developed, the so-called *figurate* numbers, had many remarkable properties. The most useful of these, and for our present purposes the most important, is the fact that this table gives the value of the expression ${}_nC_r$, for all positive integral values of n and r (including 0). The expression ${}_nC_r$ means the number of combinations of n things taken r at a time. It is also written $\binom{n}{r}$, and is equal to

$$\frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{r!}$$

or $n!/r!\,(n-r)!$, in which $r!$ is read "factorial r" and denotes the product of all the integral numbers from 1 to r inclusive. The appropriate solution for any given values of n and r is to be found in the nth line and rth column of the arithmetical triangle. See Table I.

Now ${}_nC_r$ refers to things, each of which is capable of two and only two variations, such as coins that may fall either heads or tails. But frequently we have to do with things subject to more than two variations, such as a number of signal lights each showing several colors, or a number of dice which may fall on any one of their six faces. The solutions of such cases are not to be found in the arithmetical triangle, though in every case they can be shown to be

Names of the Columns			Natural Nos.	Triangular Nos.	Pyramidal Nos.	Pentahedroidal Nos.	Hexahedroidal Nos.	Septahedroidal Nos.	Octahedroidal Nos.	Nonahedroidal Nos.	Dekahedroidal Nos.	Undekahedroidal Nos.	
Boundaries of the Figures			Corners	Edges	Surfaces	Tetrahedra	Pentahedroids	Hexahedroids	Septahedroids	Octahedroids	Nonahedroids	Dekahedroids	
		0	1	2	3	4	5	6	7	8	9	10	Sum
	0	1											1
Point	1	1	1										2
Line	2	1	2	1									4
Triangle	3	1	3	3	1								8
Tetrahedron	4	1	4	6	4	1							16
Pentahedroid	5	1	5	10	10	5	1						32
Hexahedroid	6	1	6	15	20	15	6	1					64
Septahedroid	7	1	7	21	35	35	21	7	1				128
Octahedroid	8	1	8	28	56	70	56	28	8	1			256
Nonahedroid	9	1	9	36	84	126	126	84	36	9	1		512
Dekahedroid	10	1	10	45	120	210	252	210	120	45	10	1	1024
Combinations		${}_nC_0$	${}_nC_1$	${}_nC_2$	${}_nC_3$		${}_nC_r$			${}_nC_{n-2}$	${}_nC_{n-1}$	${}_nC_n$	
General Formulas		$\frac{n!}{0!\,(n-0)!}$	$\frac{n!}{1!\,(n-1)!}$	$\frac{n!}{2!\,(n-2)!}$	$\frac{n!}{3!\,(n-3)!}$		$\frac{n!}{r!\,(n-r)!}$			$\frac{n!}{(n-2)!\,2!}$	$\frac{n!}{(n-1)!\,1!}$	$\frac{n!}{n!\,0!}$	2^n

TABLE I. The Arithmetical Triangle.
See also Table VI for another arrangement.

the product of two or more numbers there to be found. So far as the writer is aware no systematic method of selecting the proper factors has yet been given.

In the case of two variations, for any given value of n there will be $n+1$ classes, obtained by giving r successively all integral values from 0 to n. In any class r is the number of one kind present, $n-r$ the number of the other. These can all appropriately be arranged along a straight line. In fact the complete set of solutions is to be found in the nth line of the arithmetical triangle. But if the n things are capable of more than two variations—if for example they may be A's, B's, C's, D's, etc.—then a much larger number of classes arises; for to any one of these letters may be assigned in turn all the integers from 0 to n, and all vary independently. These classes cannot be so simply arranged, and the task of obtaining all of them and calculating the number of combinations for each becomes very complicated. Some systematic method must be adopted to insure exhaustive enumeration.

The object of the present paper is to show how these cases of many variations may be appropriately arranged in more-dimensional tables, so as to develop with certainty all possible classes, and show their proper relations to one another, and also to show how the arithmetical triangle may likewise be extended to more dimensions, and thus provide means of readily finding the number of combinations corresponding to each class. The method is somewhat complicated to explain, but easy to operate. We shall begin by describing a few of the many remarkable properties of the arithmetical triangle, such as will be useful to us, and then take up in turn its extension to 2, 3, 4,. . . . , k variations.

All the numbers of the arithmetical triangle can of course be calculated from the general formula already given, $n!/r!\,(n-r)!$. But the table can also be much more simply produced by a process of successive addition as follows: Beginning with 1, below any line write the same line moved one place to the right and add. The result is the next line. The process is shown in Table II.

From the mode of development it is apparent that the differences of any column are to be found in the next column to the left. Any column is therefore an arithmetical series of the rth order, whose rth differences are constant and equal to 1. The table is in fact the complete system of all arithmetical series whose final differences are 1. Conversely each number gives the sum of all the

preceding numbers of the next column to the left, or the sum of any two numbers in the same line is found immediately below the right-hand one.

Each line gives the binomial coefficients in order for the exponent corresponding to the number of the line, for these coefficients are also given by the formula $\binom{n}{r}$ The sum of all the numbers of any line is 2^n.

The columns have been given special names because of certain properties they possess. The zero column is composed only of units. The first column contains the natural numbers. The second contains the triangular numbers, so called because they give the number of units that can be arranged in a triangle, having succes-

$$\begin{array}{llllll}
\text{Line Zero} & 1 & & & & \\
 & & 1 & & & \\ \cline{2-3}
\text{Line One} & 1 & 1 & & & \\
 & & 1 & 1 & & \\ \cline{2-4}
\text{Line Two} & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\ \cline{2-5}
\text{Line Three} & 1 & 3 & 3 & 1 & \\
 & & 1 & 3 & 3 & 1 \\ \cline{2-6}
\text{Line Four} & 1 & 4 & 6 & 4 & 1
\end{array}$$

TABLE II. Method of Constructing Arithmetical Triangle.

sively 1, 2, 3, 4,... units on a side. The third column contains the pyramidal numbers, so called because they give the number of units that can be piled like cannon balls in the form of a triangular pyramid or tetrahedron, having successively 1, 2, 3, 4,... units on a side.

The remaining columns have as yet received no special names, but they might appropriately be named after the succeeding higher-dimensional pyramids, since they similarly give in turn the numbers of units that can be arranged in the form of these higher pyramids, having successively 1, 2, 3, 4,... units on a side. We shall call the latter, after Stringham, successively, the 4-dimensional pentahedroid, the 5-dimensional hexahedroid, the 6-dimensional septahedroid, etc., in general the $(k-1)$-dimensional k-hedroid, and name the columns

after them as shown in Table I. Thus the tetrahedron becomes a 4-hedroid, the triangle a 3-hedroid, the line a 2-hedroid, the point a 1-hedroid, and the corresponding columns the 4, 3, 2, 1-hedroidal numbers respectively.

Most useful for our subsequent purposes however is the fact that the arithmetical triangle gives complete specifications for the construction of any of these higher pyramids. Thus the nth line gives in order the number of 0, 1, 2, 3,.... $(n-1)$-dimensional boundaries of the $(n-1)$-dimensional n-hedroid. We have only to read the line designating the succeeding numbers in turn as so many corners, edges, surfaces, tetrahedra, etc., as indicated in Table I. Thus we may read:

First line: One point has 1 corner or 0-space boundary.

Second line: One line has 2 corners or ends, and 1 edge or interior 1-space.

Third line: One triangle has 3 corners, 3 edges, and 1 interior 2-space or surface.

Fourth line: One tetrahedron has 4 corners, 6 edges, 4 surfaces, and 1 interior 3-space.

Fifth line: One pentahedroid has 5 corners, 10 edges, 10 triangular surfaces, and 1 interior space of four dimensions.

Similarly the remaining lines may be read.

It may be noted that as the line lies between its ends, the triangle within its edges, the 3-space of the tetrahedron inside its bounding 2-space surfaces, so the 4-space of the pentahedroid is *inside* its bounding 3-space tetrahedra. Similarly with the higher pyramids. The 5-space is inside the 4-space, the 6- inside the 5-, etc. We get to higher and higher degrees of insideness.

As we shall use these higher pyramids to represent our combinations of many variations, it is important to know how they are constructed.

We may now proceed to our task of applying the arithmetical triangle to the cases of two and more variations. Calling k the number of variations, it will be found in every case that a $(k-1)$-dimensional k-hedroidal table will be required, the total number of classes is given by the $(n+1)$th k-hedroidal number, while the sum of all the combinations is k^n. This gives a valuable check on the correctness of the work. The variations we shall call A, B, C, D, etc.

Let $k=2$. The complete set of solutions for this case, as has already been stated, is to be found in the nth line of the arithmetical

triangle. They form therefore a linear table, as shown for $n=10$, in Table III. The second half of any such line is always the reverse of the first half, so that there are only $(n+2)/2$, disregarding remainder, different numbers in it.

	0	1	2	3	4	5	6	7	8	9	10	B
A	10	9	8	7	6	5	4	3	2	1	0	
	1	10	45	120	210	252	210	120	45	10	1	

TABLE III. All Classes Combinations of 10 Things each susceptible of 2 Variations, such as 10 Coins, that may fall Heads or Tails.

Number of classes $= n + 1 = 11$. Sum of combinations $= 2^{10} = 1024$.
Note. First line shows number of B's present, second line number of A's.

Let $k=3$. Each thing may now be an A or a B or a C. The additional classes possible may be derived from the previous case in the following way. To each class of A and B, as shown in Table III, additional classes may be formed, by keeping the B's constant, and exchanging the A's one by one for C's. The number of new classes thus formed will in each case be equal to the number of A's contained in the original cell. Thus the first cell gives 10, the second 9, the third 8, etc. new classes. The only appropriate way to arrange the new classes is in columns vertically under the cells from which they were developed, according to descending values of A, or ascending values of C. In this way the triangular Table IV is obtained, for the value of $n=10$. It may be noted that there will be three classes where the n things are all of one kind, as all A's, all B's, or all C's. These are placed at the corners, or 0-dimensional boundaries of the triangle. There will be three sets of classes where the things are AB, BC, or AC. Each of these three sets is a duplicate of Table III, and they are located on the three edges of the triangle. The remaining classes in which all three variations appear are all located in the interior of the triangle, or in 2-space.

Further the triangle may be divided, as shown by the dash lines of Table IV, into a series of similar concentric triangular shells. The first interior shell contains all classes where one of the letters or variations appears each time only once. The second interior shell contains all classes where one of the letters appears always twice, the third three, etc. There will always be $n/3$, disregarding remainder, of these interior shells. It may further be noted that in each line,

column and diagonal, the second half is the reverse of the first half. The same numbers are hence oft repeated. In fact in Table IV there are only 14 different numbers, while if we deduct the 6 taken direct from Table III there are only 8 new numbers to calculate. These are shown enclosed in heavy lines in the table. It is easily seen that they cover approximately 1/6 of the total area of the triangle, hence may be calculated for any n from $n^2/12$, taking the

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

Note. The first line shows number of B's present, left-hand column the number of C's. The remaining $n-(B+C)$ are A's.

Number of classes = 11th triangular number or $[(n+1)(n+2)]/2 = 66$.

Sum of all combinations $= 3^{10} = 59{,}049$.

TABLE IV. Combinations of 3 Variations for $n = 10$.

nearest whole number. For the values of n from 1 to 15 the following are obtained as the new interior numbers to calculate:

$n =$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
Nos. 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19.

The fact that for the first two cases no new numbers are obtained means that all the cells abut on the edges. When $n = 3$ we get 1 interior cell; when $n = 4$ we get 3 cells but all are filled with the same number, viz. 12, as the reader may verify, if he likes, by working out these simple cases. Hence the number of calculations to make is not very great unless n is large.

It will be noted of course that the numbers increase in size toward the geometric center of the triangle. Also from the method

of numbering, the B's are constant in any column, the C's along any line, and each increases in value from the right angle outward. The A's are constant along any diagonal and increase in value toward the right angle, where they are all A's. This method of lettering and numbering will be adhered to in all that follows, and the characteristics that depend upon it will naturally always recur.

The value of any interior number is given by the general formula $\frac{n!}{A!B!C!} = \frac{n!}{B!C![n-(B+C)]!}$, since $A+B+C=n$. Calling $0!=1$, as is customary, this formula will also apply to the edges and hence to all the numbers of the triangle. But there is a much easier way of deriving the appropriate numbers directly from the arithmetical triangle, which it is one of the objects of this paper to show. If in the above formula we give to A, B and C different values, and write them down in their proper places, we obtain Table V, below. Since, as already stated, the A's, B's and C's are constant respectively along the diagonals, columns and lines it is apparent that the expression $n!/A!B!C!$ can, in three ways, be divided into two factors, one of which is constant and the other variable, according as we choose the constant part along a diagonal, a column or a line. Thus,—

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!} \times \frac{1}{C![n-(B+C)]!} \tag{1}$$

$$\text{or } = \frac{n!}{C!} \times \frac{1}{B![n-(B+C)]!} \tag{2}$$

$$\text{or } = \frac{n!}{[n-(B+C)]!} \times \frac{1}{B!C!} \tag{3}$$

In every case the first factor is constant, the second variable. In (1) first factor is constant along a column, in (2) along a line, in (3) along a diagonal. Not much is gained by this, but if we multiply numerator and denominator in the three cases respectively by $(n-B)!$, $(n-C)!$, $(B+C)!$, we obtain,

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!(n-B)!} \times \frac{(n-B)!}{C![n-(B+C)]!} = {}_nC_B \times {}_{n-B}C_C \tag{4}$$

$$\text{or } = \frac{n!}{C!(n-C)!} \times \frac{(n-C)!}{B![n-(B+C)]!} = {}_nC_C \times {}_{n-C}C_B \tag{5}$$

$$\text{or } = \frac{n!}{(B+C)![n-(B+C)]!} \times \frac{(B+C)!}{B!C!} = {}_nC_{B+C} \times {}_{B+C}C_B \tag{6}$$

In all three cases now each factor is seen to be a figurate number, hence one to be found in the arithmetical triangle. Moreover

A	0	1	2	3	4	5	6	B
0	$\frac{n!}{0!\,0!\,[n-(0+0)]!}$	$\frac{n!}{1!\,0!\,[n-(1+0)]!}$	$\frac{n!}{2!\,0!\,[n-(2+0)]!}$	$\frac{n!}{3!\,0!\,[n-(3+0)]!}$	$\frac{n!}{4!\,0!\,[n-(4\times 0)]!}$	$\frac{n!}{5!\,0!\,[n-(5+0)]!}$	$\frac{n!}{6!\,0!\,[n-(6+0)]!}$	
1	$\frac{n!}{0!\,1!\,[n-(0+1)]!}$	$\frac{n!}{1!\,1!\,[n-(1+1)]!}$	$\frac{n!}{2!\,1!\,[n-(2+1)]!}$	$\frac{n!}{3!\,1!\,[n-(3+1)]!}$	$\frac{n!}{4!\,1!\,[n-(4+1)]!}$	$\frac{n!}{5!\,1!\,[n-(5+1)]!}$		
2	$\frac{n!}{0!\,2!\,[n-(0+2)]!}$	$\frac{n!}{1!\,2!\,[n-(1+2)]!}$	$\frac{n!}{2!\,2!\,[n-(2+2)]!}$	$\frac{n!}{3!\,2!\,[n-(3+2)]!}$	$\frac{n!}{4!\,2!\,[n-(4+2)]!}$			
3	$\frac{n!}{0!\,3!\,[n-(0+3)]!}$	$\frac{n!}{1!\,3!\,[n-(1+3)]!}$	$\frac{n!}{2!\,3!\,[n-(2+3)]!}$	$\frac{n!}{3!\,3!\,[n-(3+3)]!}$				
4	$\frac{n!}{0!\,4!\,[n-(0+4)]!}$	$\frac{n!}{1!\,4!\,[n-(1+4)]!}$	$\frac{n!}{2!\,4!\,[n-(2+4)]!}$					
5	$\frac{n!}{0!\,5!\,[n-(0+5)]!}$	$\frac{n!}{1!\,5!\,[n-(1+5)]!}$						
6	$\frac{n!}{0!\,6!\,[n-(0+6)]!}$							
C								

TABLE V.

Formulas for Calculating Combinations of 3 Variations.

This table is of indefinite Extent, but for any given value of n comes to an end when $n = B + C$.

First factorial of denominator gives number of B's.
Second " " " " " C's.
Third " " " " " A's.

General formula: $\frac{n!}{B!\,C!\,[n-(B+C)]!}$

the first factor is not only constant but in (4) it is *equal* to the number that stands at the head (and foot) of the corresponding column, in (5) to the number that stands at either end of the corresponding line, in (6) to the number that stands at either end of the corresponding diagonal of the table to be calculated, while in each case the second factor for these terms becomes 1. By the first scheme then all the members of any column could be obtained by multiplying the first one by the successive values of the second factor of (4), obtained by giving B and C the proper values. Similarly by the second scheme all the terms of any line could be obtained from the first one, while by the third scheme all the terms of any diagonal from the end ones. Any one of the three schemes would produce the whole triangle, and it would seem natural to choose either of the first two. However because of the occurrence of $n-\mathrm{B}$, and $n-\mathrm{C}$, in these two schemes, making it necessary to assign a definite value to n before anything can be done, they do not lend themselves so readily to general treatment as the third scheme. We shall accordingly adopt the latter.

The proper values of the second factor, or coefficient as we shall call it, could of course be found in line B, or C, of column $\mathrm{B}+\mathrm{C}$ of the arithmetical triangle. But if we give different values to B and C and write the resulting numbers down in their proper places in the triangle, we obtain Table VI. It is at once seen that they follow a very regular order, being in fact nothing other than the arithmetical triangle itself, with each column pushed up to the top. This is the usual arrangement of the figurate numbers. The diagonals of the new table are the lines of the old.

The procedure of calculating a triangle then is as follows. As the first line write the nth line of the arithmetical triangle. Any interior number is then calculated by multiplying the number that stands at the end of its diagonal by the coefficient shown in Table VI. The process is shown in Table VII. Or the process may be described thus: Each successive line is derived from the first line by discarding each time one additional term and multiplying the remaining terms first by the natural, then by the triangular, then by the pyramidal, etc., numbers in order. Or we may sum the whole thing up in one general rule:

The mth *line of a "surface triangle" is derived from the* nth *line of the arithmetical triangle by discarding* $m-1$ *terms and multi-*

plying the remaining terms by the m-hedroidal numbers in order, beginning with the first one.

This rule applies to the outside edge as well as to the inner lines. The reason for calling the figure a *surface triangle* is because the surfaces of all our subsequent higher pyramids will be composed of such figures.

We have used for purposes of illustration $n=10$. If we give to n other values we shall obtain similar triangles, smaller or larger according to the value of n. If we construct a series of these from $n=0$ up, and pile them all up on top of each other with their A

A	0	1	2	3	4	5	6	7	8	9	10	11	12	B
0	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	2	3	4	5	6	7	8	9	10	11	12		Natural Nos.
2	1	3	6	10	15	21	28	36	45	55	66			Triangular Nos.
3	1	4	10	20	35	56	84	120	165	220				Pyramidal Nos.
4	1	5	15	35	70	126	210	330	495					Pentahedroidal Nos.
5	1	6	21	56	126	252	462	792						Hexahedroidal Nos. Etc.
6	1	7	28	84	210	462	924							
7	1	8	36	120	330	792								
8	1	9	45	165	496									
9	1	10	55	220										
10	1	11	66											
11	1	12												
12	1													
C														

TABLE VI.

Coefficients for Calculating Combinations of 3 Variations or General Table of Figurate Numbers.

vertices coinciding, we shall obtain the *arithmetical pyramid* of three dimensions, as shown in Fig. 1. This has the form of a number of cubes piled in a corner of the room. It can of course be indefinitely extended. In it will be found all classes of combinations for $k=3$, just as in the plane arithmetical triangle are to be found all classes for $k=2$. Each surface will be an arithmetical of the old sort, all the new classes of three variations being found in the interior cells.

As the numbers ${}_nC_r$ are binomial coefficients in the expansion of $(a+b)^n$, so the numbers representing combinations of three

variations are the trinomial coefficients arising in the expansion of $(a+b+c)^n$. Thus Table IV enables us at once to write out the expansion of $(a+b+c)^{10}$. In like manner the numbers representing combinations of k variations are k-nomial coefficients arising in the expansion of the nth power of a polynomial of k terms.

Let $k=4$. Each thing may now be an A or a B or a C or a D. In order to develop and represent all possible classes in their proper relations to each other, use will now have to be made of a three-

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1 ×1	10 ×1	45 ×1	120 ×1	210 ×1	252 ×1	210 ×1	120 ×1	45 ×1	10 ×1	1 ×1	
1	10 ×1	45 ×2	120 ×3	210 ×4	252 ×5	210 ×6	120 ×7	45 ×8	10 ×9	1 ×10		
2	45 ×1	120 ×3	210 ×6	252 ×10	210 ×15	120 ×21	45 ×28	10 ×36	1 ×45			
3	120 ×1	210 ×4	252 ×10	210 ×20	120 ×35	45 ×56	10 ×84	1 ×120				
4	210 ×1	252 ×5	210 ×15	120 ×35	45 ×70	10 ×126	1 ×210					
5	252 ×1	210 ×6	120 ×21	45 ×56	10 ×126	1 ×252						
6	210 ×1	120 ×7	45 ×28	10 ×84	1 ×210							
7	120 ×1	45 ×8	10 ×36	1 ×120								
8	45 ×1	10 ×9	1 ×45									
9	10 ×1	1 ×10										
10	1 ×1											
C												

TABLE VII.

Method of Calculating Combinations for $n=10$, $k=3$, from 10th Line of Arithmetical Triangle, and Coefficients of Table VI. Result is Table IV.

Only the 8 numbers enclosed in heavy lines need be calculated, all the others being taken direct or repetitions.

dimensional arrangement. For to every class containing A, B and C, as shown in Table IV, additional classes may now be formed by exchanging the A's successively for D's. The only proper place to put these new classes is to build them out in the third dimension from the original cells of Table IV from which they were developed. The resulting arrangement will be pyramidal in form because of the regularly decreasing value of A from the corner outward to the last diagonal. Such a pyramid, as we read in the fourth line of the arithmetical triangle, will be composed of 4 corners, 6 edges,

4 surfaces and 1 interior space. These will carry respectively the classes where 1, 2, 3 and 4 variations are present. There will be the following groups of such classes:

1 Variation,	A, B, C, D,	$= {}_4C_1, = 4$
2 Variations,	$\left\{\begin{matrix} AB & BC & CD \\ AC & BD & \\ AD & & \end{matrix}\right\}$	$= {}_4C_2, = 6$
3 Variations,	ABC ABD ACD BCD	$= {}_4C_3, = 4$
4 Variations,	ABCD	$= {}_4C_4, = 1$

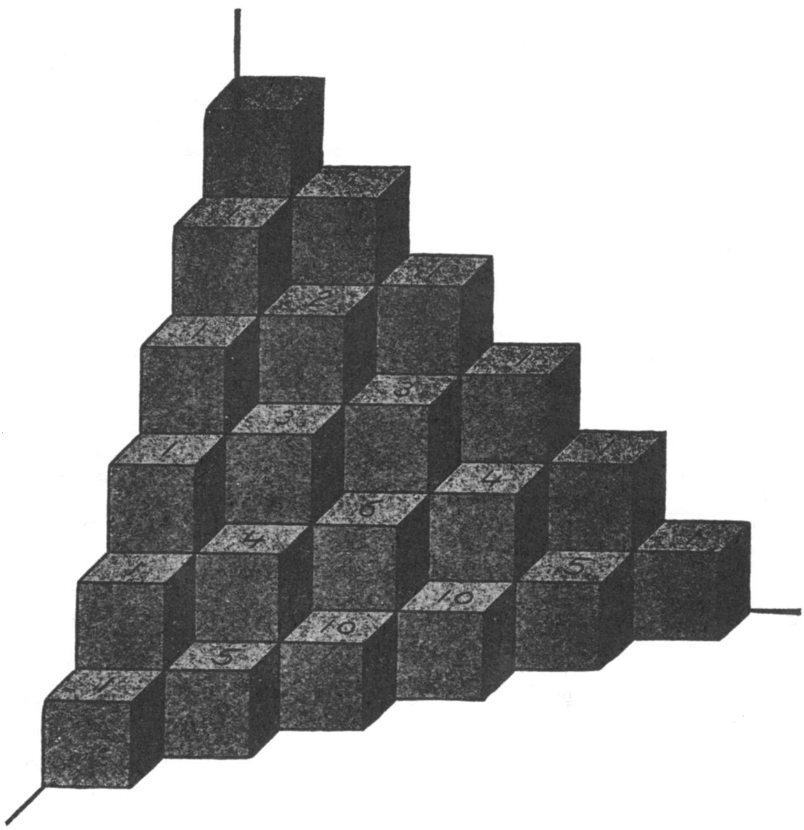

Fig. 1.

Each of the four surfaces of the pyramid will, for $n = 10$, be exactly the same as the triangle of Table IV, only as the 12 edges

of the 4 triangles coincide in pairs, thus reducing to 6, and as the 12 corners coincide in threes, thus reducing to 4, we cannot simply repeat the whole triangle four times, but regard must be had for the corners and edges to be omitted. One way would be to represent the corners, edges and interior part of the triangles separately. Another method is shown in Fig. 2. Here the slant sides are supposed to be folded down into the plane of the base, so as to depict all in one plane. The triangle BCD, which is in reality equilateral, is conveniently made right-angled like the others. The sides to be omitted are indicated by dotted lines. A modification of this plan, that economises space, is shown in Fig. 3. A modification of the first method is used in Diag. 1, Table VIII, and in most of the

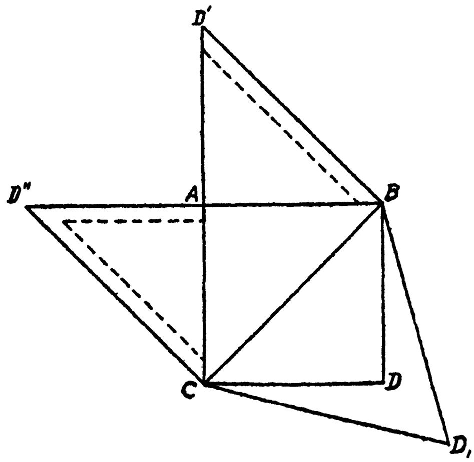

Fig. 2.

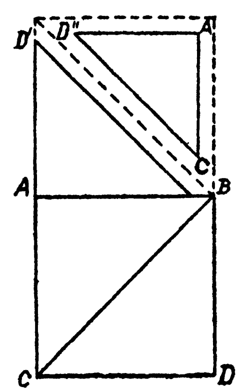

Fig. 3.

higher pyramids. The method of Fig. 2 is used in Diag. 5, Table XII, that of Fig. 3 in Diags. 2 and 3 of Table VIII.

The surface of the pyramid being thus represented, and no new numbers to calculate for it, it remains to consider only the representation and calculation of the interior cells, where four variations are present. To do this it will be found most convenient to consider the pyramid as made up of a number of concentric pyramidal shells, each one cell in thickness, like, to use an unsavory simile, the coatings of an onion. Each interior shell will then be exactly similar to the surface shell, which has just been pealed off. It can be represented in the same way, by any one of the three methods described. Analogous to the triangular shells of Table IV,

each cell of the first inner pyramidal shell will contain, beside the numbers indicated of the letters that stand at the vertices of the particular triangle in which it is found, one example of the missing letter. Thus the four triangles of the first inner shell are appropriately lettered, ABC+1D, ABD +1C, ACD+1B, DCB+1A, as in

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

TABLE VIII, DIAG. 1.
Typical Surface Triangle of Pyramid for $n=10$, $k=3$.

The pyramid has 4 corners, 6 edges, 4 surfaces.

Each corner has $1\times 4 = 4$ or k classes.
" edge " $9\times 6 = 54$ " $6(n-1)$ "
" surface " $36\times 4 = 144$ " $4\frac{(n-1)n}{2}$ " ; or $(n-1)$th Triangular No.

Total number of classes = 202.

Each corner has $1\times 4 =$ 4 or k Combinations
Each edge " $1{,}022\times 6 =$ 6,132 or $6(2^n-2)$ "
Each inner surface " $55{,}980\times 4 = 223{,}920$ or $4[3^n-3(2^n-2)-3]$ "

Total comb. of surface shell = 230,056.

The four triangles are to be lettered, ABC, ABD, ACD, DCB, as in Figs. 2 and 3.

Diag. 2, Table VIII. Similarly the cells of the second inner shell contain two of the missing letter, and are lettered as in Diag. 3, Table VIII. The third shell, when there is one, will contain 3, the fourth 4, etc., of the missing letter. Each inner shell will have four less cells on a side than the next outer one. Hence until $n=4$

	D			D	5	4	3	2	A	
	6	2520			7560	12600	12600	7560	2	
	5	5040	7560			12600	16800	12600	3	
ABD + 1C	4	6300	12600	12600			12600	12600	4	ACD + 1B
	3	5040	12600	16800	12600			7560	5	
	2	2520	7560	12600	12600	7560			C	
	A	1	2	3	4	5	6	7	B	
	1	720	2520	5040	6300	5040	2520	720	7	
	2	2520	7560	12600	12600	7560	2520	2520	6	
	3	5040	12600	16800	12600	5040	7560	5040	5	
ABC + 1D	4	6300	12600	12600	6300	12600	12600	6300	4	DCB + 1A
	5	5040	7560	5040	12600	16800	12600	5040	3	
	6	2520	2520	7560	12600	12600	7560	2520	2	
	7	720	2520	5040	6300	5040	2520	720	1	
	C	7	6	5	4	3	2	1	D	

TABLE VIII, DIAG. 2.
First Inner Shell of Pyramid.
No. of classes = 74. No. of comb. = 591,720. 7 Nos. to calculate.

	D		D		A	
ABD + 2C						ACD + 2B
	3	25200			C	
	A	2	3	4	B	
	2	18900	25200	18900	4	
ABC + 2D	3	25200	25200	25200	3	DCB + 2 A
	4	18900	25200	18900	2	
	C	4	3	2	D	

TABLE VIII, DIAG. 3.
Second Inner Shell of Pyramid.
No. of classes = 10. No. of comb. = 226,800. 2 Nos. to calculate.

Summary of Pyramid.

4	Classes of 1 Variation	having	4	Combinations.
54	" " 2 Variations	"	6,132	"
144	" " 3 "	"	223,920	"
84	" " 4 "	"	818,520	"
286	" " all "	"	1,048,576	"
	= 11th Pyramidal Number.		$= 4^{10}$.	

there will be no inner shells. The number of inner shells will always be $n/4$ disregarding the remainder. The first inner shell will be numbered from 1 to $n-3$, the second from 2 to $n-2\cdot 3$, the third from 3 to $n-3\cdot 3$; in general, if t is the number of the shell it will be numbered from t to $n-3t$. These inner shells we shall call for short the first tetra, second tetra, etc. They are not really tetrahedral in shape, being composed mostly of right-angled, instead of equilateral triangles, bút it will be convenient to call them as designated.

In Table VIII is worked out this case for $n=10$. Diag. 1, which represents the typical surface triangle ABC, is the same as in Table IV. Only the interior portion, enclosed in heavy lines, is exactly repeated on the other three surfaces. The edges, as stated, are repeated six, the corners four times. In reading the triangles it must be remembered that the numbers along the edges refer to the letters that stand at the acute angles. For the inner shells one or two of the missing letter are to be added, and the remainder of the n things are then of the letter that stands at the right angle.

The general formula for calculating the number of combinations corresponding to any interior cell is $n!/A!\,B!\,C!\,D!$. We need only consider in each case the typical triangle $ABC+tD$. Since for the first tetra, $D=1$, the general formula becomes $\dfrac{n!}{B!\,C!\,1!\,[n-(B+C+1)]!}$. If we give various values to B and C, and put them in their proper places in the triangle, we obtain Table IX. It is apparent that this table has all the same regularities as Table V, so that we could here also obtain the interior terms from the edge terms of either columns, lines or diagonals, by determining proper coefficients. But we do not yet know the edge terms. These must themselves be derived somehow. If in imagination we follow any diagonal of the tetra out beyond the latter to where it pierces the surface of the pyramid, we shall find that it ends in a term that is suitable for our calculation. This feat of the imagination may not seem so easy, but the following plan may help.

Suppose Table V composed of a horizontal layer of cubes. Then Table IX, also composed of a horizontal layer of cubes, is to be set down on top of it, so that its first term lies upon the second term,

A	1	2	3	4	5	6	B
1	$\frac{n!}{1!\,1!\,1!\,(n-3)!}$	$\frac{n!}{2!\,1!\,1!\,(n-4)!}$	$\frac{n!}{3!\,1!\,1!\,(n-5)!}$	$\frac{n!}{4!\,1!\,1!\,(n-6)!}$	$\frac{n!}{5!\,1!\,1!\,(n-7)!}$	$\frac{n!}{6!\,1!\,1!\,(n-8)!}$	
2	$\frac{n!}{1!\,2!\,1!\,(n-4)!}$	$\frac{n!}{2!\,2!\,1!\,(n-5)!}$	$\frac{n!}{3!\,2!\,1!\,(n-6)!}$	$\frac{n!}{4!\,2!\,1!\,(n-7)!}$	$\frac{n!}{5!\,2!\,1!\,(n-8)!}$		
3	$\frac{n!}{1!\,3!\,1!\,(n-5)!}$	$\frac{n!}{2!\,3!\,1!\,(n-6)!}$	$\frac{n!}{3!\,3!\,1!\,(n-7)!}$	$\frac{n!}{4!\,3!\,1!\,(n-8)!}$			
4	$\frac{n!}{1!\,4!\,1!\,(n-6)!}$	$\frac{n!}{2!\,4!\,1!\,(n-7)!}$	$\frac{n!}{3!\,4!\,1!\,(n-8(!}$				
5	$\frac{n!}{1!\,5!\,1!\,(n-7)!}$	$\frac{n!}{2!\,5!\,1!\,(n-8)!}$					
6	$\frac{n!}{1!\,6!\,1!\,(n-8)!}$						
C							

TABLE IX.

Formulas for calculating Typical Triangle ABC+1D of First Inner Pyramidal Shell.

General Formula $\frac{n!}{B!\,C!\,1!\,[n-(B+C+1)]!}$

Triangle is numbered from 1 to n-3. Table ends when n=B+C+1.

second column, of Table V. The latter now twice repeated, but with one line removed, is to be supposed set up on edge so as to enclose the right angle of IX with two vertical walls. Then it

is not difficult to see that the diagonal of the first term of IX pierces these back walls in the third term of the second line. The next diagonal of IX, of course, meets the fourth term, etc. By

A	1	2	3	4	5	6	7	8	B
1	2	3	4	5	6	7	8	9	Natural Nos.
2	3	6	10	15	21	28	36	Triangular Nos.	
3	4	10	20	35	56	84	Pyramidal Nos.		
4	5	15	35	70	126		Etc.		
5	6	21	56	126					
6	7	28	84						
7	8	36							
8	9								
C									

TABLE X.

Coefficients for Calculating Typical Triangle of First Inner Pyramidal Shell.

Table is of Indefinite Extent.

A	1	2	3	4	5	6	7	B
1	2× 360	3× 840	4× 1260	5× 1260	6× 840	7× 360	8× 90	
2	3× 840	6× 1260	10× 1260	15× 840	21× 360	28× 90		
3	4× 1260	10× 1260	20× 840	35× 360	56× 90			
4	5× 1260	15× 840	35× 360	70× 90				
5	6× 840	21× 360	56× 90					
6	7× 360	28× 90						
7	8× 90							
C								

TABLE XI.

Method of Calculating Triangle ABC of Diag. 2, Table VII, from Second Line of Surface Triangle and Coefficients of Table X.

comparison of the two tables it is in fact seen that the factor $n!/[n-(B+C+1)]!$, which is constant along the diagonals of IX, is the same in the surface terms in which they end. It is only necessary then to determine the proper coefficients. This can be done as before by factoring the general expression, thus:

$$\frac{n!}{B!\,C!\,1!\,[n-(B+C+1)]!} =$$

$$\frac{n!}{(B+C)!\,1!\,[n-(B+C+1)]!} \times \frac{(B+C)!}{B!\,C!} = {}_nC_{B+C} \times {}_{B+C}C_B$$

Giving B and C their various values as before, we obtain Table X of the coefficients. It is seen at once that this table is exactly the same as Table VI for the surface triangle, except that the first line and column are omitted. The calculation of these inner terms hence becomes extremely simple, and may be reduced to the following rule.

The *m*th line of the first tetra is derived from the 2d line of the surface triangle by discarding $m+1$ terms, and multiplying the remaining terms by the $(m+1)$-hedroidal numbers in order, beginning with the second.

A similar investigation will lead to a similarly simple result for the second tetra, which may be reduced to the following rule:

The *m*th line of the second tetra is derived from the 3d line of the surface triangle, by discarding $m+3$ terms, and multiplying the remaining terms by the $(m+2)$-hedroidal numbers in order, beginning with the third.

Similar rules may be derived for the succeeding tetra, but if we call t the number of the tetra we may combine them all in one general rule as follows:

The m*th line of the* t*th tetra is derived from the* $(t+1)$th *line of the surface triangle by discarding* $2t+m-1$ *terms and multiplying the remaining terms by the* $(t+m)$*-hedroidal numbers in order, beginning with the* $(t+1)$th.

This rule is general not only for all the inner tetras, but by putting t in it equal to 0 it reduces to the rule previously given for the surface triangle, which thus may be considered as the 0-tetra. This one rule hence covers all cases up to the present.

If we construct a series of pyramids, like that of Table VIII, for the successive values of n from 0 up, but give each a thickness of one cell in the direction of the *fourth* dimension, and pile the successive pyramids so that their A vertices are adjacent to each other in the direction of this dimension, then we shall obtain the four-dimensional arithmetical pyramid. Each three-dimensional pyramid will be a slice of the four-dimensional one, perpendicular to its fourth-dimensional axis, just as each two-dimensional diagram of Fig. 1 is a slice of the three-dimensional pyramid. Each cubical cell will now acquire a thickness equal to its edge in the direction of the fourth dimension and so become a four-dimensional cube, or teseract as it is sometimes called. The whole system will of course contain all classes of combinations up to four variations.

II.

In Part I we have dealt with the combinations of any number of things, each capable of 1, 2, 3 or 4 variations, and found that all possibilities could be represented by tables, having respectively 0, 1, 2 and 3 dimensions, viz., by the point, line, triangle and triangular pyramid. In each case we required a table of $k-1$ dimensions. Hence if we allow more than four variations we must, by the same rule, step out into space of higher dimensions, making use in each case of a $(k-1)$-dimensional pyramid.

Let us first take the case of $k=5$. Call the variations A, B, C, D and E. By reasoning exactly analogous to that of the case $k=4$, it is clear that from every ABCD cell of the three-dimensional pyramid can be developed a series of new cells equal to the number of A in that cell, by exchanging successively the A's for E's. The only proper place to put these new cells is to build them out from the respective ABCD cells from which they were developed, in the direction of the *fourth* dimension. Because of the regularly diminishing number of the A's in the cells, in passing outward from the A vertex toward the BCD plane, the new solid developed will have the form of a four-dimensional pyramid, analogous to the three-dimensional pyramid previously described. We shall call it a *pentahedroid,* or a *penta* for short, though it is really right-angled instead of equilateral. This penta, as shown by the fifth line of the arithmetical triangle, is bounded by 5 corners, 10 edges, 10 triangular surfaces and five tetrahedra, all enclosing an interior four-dimensional space. These configurations will carry respectively the classes of 1, 2, 3, 4 and 5 variations. The typical triangles of the classes of 1, 2, 3 and 4 variations will be exactly the same as before, except for the different number of repetitions. The five bounding tetras will have interior shells exactly the same as those of diagrams 2 and 3 of Table VIII, and these being independent of one another will be repeated in entirety five times. The 20 surface triangles of the 5 tetras however coincide in pairs, reducing to 10; the 30 edges coincide in threes, reducing to 10; the 20 corners coincide in fours, becoming 5, as already stated. In other words 4 instead of 3 edges now radiate from every vertex, 3 instead of 2 planes from every edge, while every plane divides 2 adjacent tetras from each other.

These 10 surface triangles and the interior shells of the 5

bounding tetras constitute the surface or zero shell of the pentahedroid. The interior space can be considered as before to be made up of concentric pentahedroidal shells, each one cell in thickness in the direction of the fourth dimension. Each such shell will be exactly similar to the surface shell. It will have the same number and kind of boundaries, and can hence be represented in just the same way, viz., by 10 surface triangles, and the interior shells of the 5 bounding tetrahedra. The latter will be called: the first inner tetra shell of the first inner penta shell, second tetra of first penta, etc.

Each inner penta shell will have five less cells on a side than the next outer shell. There will therefore be $n/5$, neglecting remainder, such inner shells. Each will contain one more of each of the two missing letters. The typical triangles, which we shall call the surface triangles of the inner pentas, will be lettered and numbered as follows:

1st penta,	ABC + 1D + 1E	1 to $n-4$
2d "	ABC + 2D + 2E	2 to $n-2\cdot 4$
3d "	ABC + 3D + 3E	3 to $n-3\cdot 4$
pth "	ABC + pD + pE	p to $n-4p$

The tetras of the inner pentas will be lettered and numbered as follows:

NAME OF TETRA	LETTERING	NUMBERING	NO. OF INNER TETRA SHELLS
1st tetra of 1st penta	ABC+2D+1E	2 to $n-3-4$	$\frac{n-5}{4}$
2d " " 1 "	ABC+3D+1E	3 to $n-2\cdot3-4$	
tth " " 1 "	ABC+$(t+1)$D+1E	$t+1$ to $n-3t-4$	
1st " " 2d "	ABC+3D+2E	3 to $n-3-2\cdot4$	$\frac{n-2\cdot5}{4}$
2d " " " "	ABC+4D+2E	4 to $n-2\cdot3-2\cdot4$	
tth " " " "	ABC+$(t+2)$D+2E	$t+2$ to $n-3t-2\cdot4$	
tth " " pth "	ABC+$(t+p)$D+pE	$t+p$ to $n-3t-4p$	$\frac{n-5p}{4}$

Of course for the other triangles all combinations of the five letters will be taken. This case for $n = 10$ is worked out in Table XII. Similar tables can be made for other values of n.

Table XII.

Pentahedroidal Pyramid for $n = 10$, $k = 5$.

(Four Dimensions.)

Boundaries: 5 Corners, 10 Edges, 10 Surfaces, 5 Tetrahedra.

Diag. 1. Surface Penta Shell, Surface Triangles.

Typical Triangle ABC, same as Diag. 1, Table VIII, but

5 Corners	$\times$ 1 cell each	=	5 Cells.
10 Edges	$\times$ 9 cells "	=	90 "
10 Surfaces	$\times$ 36 cells "	=	360 "
Total Surface Cells of Surface Shell			455 "

Each Corner	has	$1 \times 5 =$	5	Combinations.
" Edge	"	$1{,}022 \times 10 =$	10,220	"
" Surface	"	$55{,}980 \times 10 =$	559,800	"
Total of Surface Triangles		=	570,025	"

The Ten Triangles are lettered,

ABC	ACD	ADE	BCD	BDE	CDE
ABD	ACE		BCE		
ABE					

Diag. 2. First Inner Tetra Shell of Surface Penta Shell.

Typical Triangle ABC + 1D, same as in Diag. 2, Table VIII.

The Shell contains 5 such Pyramids, hence

$5 \times 74 = 370$ Cells, and $5 \times 591{,}720 = 2{,}958{,}600$ Combinations.

The Five Pyramids are to be lettered,

ABCD ABCE ABDE ACDE BCDE

Each Pyramid is composed of 4 Triangles, making 20 in all for the Shell. Those of the first Pyramid ABCD are lettered,

ABC + 1D, ABD + 1C, ACD + 1B, BCD + 1A.

Similarly the other 4 Pyramids are lettered.

Diag. 3. Second Inner Tetra Shell of Surface Penta.

Typical Triangle ABC + 2D, same as Diag. 3, Table VIII.

This Diag. five times repeated gives,

$5 \times 10 = 50$ Cells, and $5 \times 226{,}800 = 1{,}134{,}000$ Combinations.

Lettering same as for first Shell.

Total of Inner Tetra Shells ....4,092,600 Comb.
Total of entire Surface Penta Shell, ..4,662,625 "

Table XII (Continued).

A	1	2	3	4	5	6	B
1	5040	15120	25200	25200	15120	5040	
2	15120	87800	50400	37800	15120		
3	25200	50400	50400	2500			
4	25200	37800	25200				
5	15120	15120					
6	5040						
C							

Table XII, Diag. 4.
First Inner Penta Shell.
Typical Surface Triangle
ABC+1D+1E

5 Corners	× 1 Cell each	=	5 Cells
10 Edges	× 4 Cells "	=	40 "
10 Surfaces	× 6 Cells "	=	60 "

(Shown within the heavy lines)
Total = 105 "

5 Corners with	5,040	Combinations each	=	25,200 Comb.
10 Edges "	80,640	" "	=	806,400 "
10 Surfaces "	264,600	" "	=	2,646,000 "
Total, of Surface Triangles, First Inner Penta			=	3,477,600 "

Lettering.

The Ten Triangles of the First Inner Penta Shell are to be lettered thus:

ABC+1D +1E	ACD+1B +1E	ADE+1B +1C
ABD+1C +1E	ACE+1B +1D	
ABE+1C +1D		
BCD+1A +1E	BDE+1A +1C	CDE+1A +1B
BCE+1A +1D		

		D			
		3			
D	3	A	2	3	B
		2	75600	75600	3
		3	75600	75600	2
		C	3	2	D

Diag. 5.

First Inner Tetra Shell of First Inner Penta Shell. Typical Triangle ABC+2D+1E.

Complete Diag. according to Fig. 2 Text.

The 5 Tetra contain 4×5=20 Cells and 75600×4×5=1,512,000 Comb.

No further Tetra Shells to the First Inner Penta.

Triangles are lettered ABC+2D+1E, DCB+2A+1E;
ACD+2B+1E, ABD+2C+1E are empty.

Similarly the other Four Tetras.

TABLE XII (Concluded).

A	2	B
2	113400	
C		

DIAG. 6.

Second Inner Penta Shell. Typical Triangle ABC+2D+2E.

This Shell has but One Cell (since $n/5=2$) and contains 2A, 2B, 2C, 2D, 2E.

Hence Total Combinations of Second Penta Shell = $10!/(2!)^5=113,400$.
Total of Inner Pentas (Diags. 4, 5 and 6) =5,103,000 Comb.

Summary of Pentahedroid.

5	Classes	of	1 Variation	having	5	Combinations.
90	"	"	2 Variations	"	10,220	"
360	"	"	3 "	"	559,800	"
420	"	"	4 "	"	4,092,600	"
126	"	"	5 "	"	5,103,000	"
1001	"	"	all "	"	9,765,625	"
	= 11th Pentahedroidal Number.					$=5^{10}$.

The general formula for any interior cell of the penta is $n!/A!B!C!D!E!$. For the surface triangles of the first inner penta this reduces to $\frac{n!}{B!C!1!1![n-(B+C+2)]!}$. Assigning B and C their various values, we obtain Table XIII, for the typical triangle, lettered ABC+1D+1E. Now the typical triangle of the first inner tetra of the surface pyramids is lettered ABC+1D, and its general formula is $\frac{n!}{B!C!1![n-(B+C+1)]!}$. It differs from the present only in the one E lacking. If we compare Table IX with XIII it is seen that the second column of the former coincides with the first column of the latter in the number of A's present in each cell, as shown by the last factor of the denominator. Further it is readily seen that every term of the latter is just double the corresponding term of the former. Comparing further the second column of XIII with the third of IX we find that every term of the former is treble the corresponding term of the latter, and so on for the following columns. We may therefore reduce this case to the following rule:

The surface triangle of the first inner penta shell is derived from the first inner tetra of the surface shell by discarding one column and multiplying the remaining columns successively by the natural numbers in order beginning with the second.

Going through a similar process for the surface triangles of the second penta, we should find that these are derived from the second tetra of the surface shell by discarding 2 columns and multiplying the remaining columns by the triangular numbers in order beginning with the third.

A	1	2	3	4	5	B
1	$\frac{n!}{1!\,1!\,1!\,1!\,(n-4)!}$	$\frac{n!}{2!\,1!\,1!\,1!\,(n-5)!}$	$\frac{n!}{3!\,1!\,1!\,1!\,(n-6)!}$	$\frac{n!}{4!\,1!\,1!\,1!\,(n-7)!}$	$\frac{n!}{5!\,1!\,1!\,1!\,(n-8)!}$	
2	$\frac{n!}{1!\,2!\,1!\,1!\,(n-5)!}$	$\frac{n!}{2!\,2!\,1!\,1!\,(n-6)!}$	$\frac{n!}{3!\,2!\,1!\,1!\,(n-7)!}$	$\frac{n!}{4!\,2!\,1!\,1!\,(n-8)!}$		
3	$\frac{n!}{1!\,3!\,1!\,1!\,(n-6)!}$	$\frac{n!}{2!\,3!\,1!\,1!\,(n-7)!}$	$\frac{n!}{3!\,3!\,1!\,1!\,(n-8)!}$			
4	$\frac{n!}{1!\,4!\,1!\,1!\,(n-7)!}$	$\frac{n!}{2!\,4!\,1!\,1!\,(n-8)!}$				
5	$\frac{n!}{1!\,5!\,1!\,1!\,(n-8)!}$					
C						

TABLE XIII.

Formulas for Calculating First Inner Penta Shell.

Typical Surface Triangle ABC+1D+1E

General Formula $\frac{n!}{B!\,C!\,1!\,1!\,[n-(B+C+2)]!}$

Table Ends when $n=B+C+2$.

Finally in general we should find:

The surface triangle of the pth penta is derived from the pth surface tetra by discarding the first p columns and multiplying

the remaining columns by the $(p+1)$-hedroidal numbers in order, beginning with the $(p+1)$th.

Examining similarly the inner tetras of the penta shells we find that the first tetra of the first penta is lettered ABC + 2D + 1E, while the second tetra of the surface is lettered ABC + 2D, differing again only by the one E lacking. Hence the former may be derived from the latter in a manner similar to the surface triangles of the pentas. Without going through all the details it may at once be stated that the following general rules may easily be derived:

The tth tetra of the *first* inner penta is derived from the $(t+1)$th surface tetra by discarding the first column and multiplying the remaining columns by the natural numbers in order beginning with the $(t+2)$d.

The tth tetra of the *second* inner penta is derived from the $(t+2)$d surface tetra by discarding two columns and multiplying the remaining columns by the triangular numbers in order, beginning with the $(t+3)$d.

Finally we may set up the following perfectly general rule for any tetra:

The tth tetra of the pth penta is derived from the $(t+p)$ surface tetra by discarding the first p columns and multiplying the remaining columns successively by the $(p+1)$-hedroidal numbers in order, beginning with the $(t+p+1)$th.

Substituting in the above $t=0$, we get the rule for the surface triangles of any inner penta, previously given, so that this rule is perfectly general for all inner pentas and their attached tetras.

Finally let $k=6$, viz., A, B, C, D, E and F. This is the case which is presented by a number of dice, each one of which may fall on any one of its six faces. We shall now require for proper representation of all classes a *five*-dimensional pyramid, or hexahedroid, or hexa as we shall call it for short. From the sixth line of the arithmetical triangle we find that such a figure is bounded by 6 corners, 15 edges, 20 surfaces, 15 tetrahedra, 6 pentahedra, and contains one interior five-dimensional space. These will carry the classes of 1, 2, 3, 4, 5 and 6 variations respectively. The classes of 1, 2 and 3 variations will be represented by the 20 surface triangles, each exactly the same as the previous cases except that regard must be had for the proper number of repetitions of the edges and corners. The classes of 4 variations will be represented by the proper number of surface tetra shells, exactly similar to

NAME OF BOUNDARY	LETTERING	NUMBERING	NO. OF SHELLS	TRIANGLES PER SHELL
Surface Triangles of 1st Hexa	ABC+1D+1E+1F	1 to $n-5$		20
tth Tetra " " "	ABC+tD+1D+1F	$t+1$ " $n-3t-5$	$t=\frac{n-6}{4}$	$15\times4=60$
Surface of 1st Penta " " "	ABC+2D+2D+1F	2 " $n-4-5$		$6\times10=60$
tth Tetra" " " " " "	ABC+$(t+2)$D+2E+1F	$t+2$ " $n-3t-4-5$	$t=\frac{n-5-6}{4}$	$6\times5\times4=120$
Surface " pth " " " "	ABC+$(p+1)$D+$(p+1)$E+1F	$p+1$ " $n-4p-5$	$p=\frac{n-6}{5}$	$6\times10=60$
tth Tetra" " " " " "	ABC+$(t+p+1)$D+$(p+1)$E+1F	$t+p+1$ " $n-3t-4p-5$	$t=\frac{n-5p-6}{4}$	$6\times5\times4=120$
Surface Triangles " h "	ABC+hD+hE+hF	h " $n-h$	$h=\frac{n}{6}$	20
tth Tetra " " "	ABC+$(t+h)$D+hE+hF	$t+h$ " $n-3t-5h$	$t=\frac{n-6h}{4}$	$15\times4=60$
Surface of pth Penta " " "	ABC+$(p+h)$D+$(p+h)$E+hF	$p+h$ " $n-4p-5h$	$p=\frac{n-6h}{5}$	$6\times10=60$
tth Tetra" " " " " "	ABC+$(t+p+h)$D+$(p+h)$E+hF	$t+p+h$ " $n-3t-4p-5h$	$t=\frac{n-5p-6h}{4}$	$6\times5\times4=120$

TABLE XIV. Lettering and Numbering of a Hexahedroid.

Diagrams 2 and 3 of Table VIII but 15 times repeated for the 15 bounding tetras. The classes of 5 variations will be represented by the proper number of surface penta shells, with their accompanying inner tetra shells, exactly similar to Diagrams 4, 5 and 6 of Table XII, but each 6 times repeated for the 6 bounding pentahedroids. Hence it remains only to consider the cells of the interior five-dimensional space. As before, we shall consider the interior to be made up of concentric inner hexa shells, each one cell in thickness in the direction of the fifth dimension. Each of these inner shells will have the same number and kind of boundaries as the surface or zero hexa just described, and will therefore be represented by the same series of diagrams, viz., 20 surface triangles, with their 15 edges and 6 corners, 15 surface tetra, with their inner shells, 6 bounding pentas, each in turn represented as in Table XII, by 10 surface triangles, with their 10 edges and 5 corners, and by five bounding tetras with their inner shells. Each inner hexa will have six cells less on a side than the next outer one. The number of such inner hexa shells will therefore be $n/6$, neglecting remainder.

The lettering and numbering of the surface triangles, tetras and pentas, with the tetras of the latter, is the same as in the previous case. Hence it only remains to show the numbering and lettering of the inner hexa shells. This is done in Table XIV. The whole table can be developed from the general formulas of the last line by substituting the proper values of t, p and h. In fact by substituting 0 for any of them these formulas will give the surface configurations, and hence the pentahedroid of the previous case. For example, if we put them all equal to zero we get that the surface triangle is lettered ABC and numbered from 0 to n. Also it will have $n/3$ inner triangular shells.

It remains to consider how the combinations for each interior class may be calculated. Without going through the details we may at once state that a perfectly general rule may be set up as follows:

The tth tetra of the pth penta of the hth hexa is derived from the $(t+p)$ tetra of the hth penta by discarding the first $(p+h)$ columns and multiplying the remaining columns successively by the $(p+h+1)$-hedroidal numbers in order, beginning with the $(t+p+h+1)$th.

Diag No.	Values to Assign t	p	h	r	Name of Shell	Lettering	Numbering	Derive From Shell	Dis-card	Multiply Remai'g by	Begin With	Number of Inner Shells
					tth Tetra of pth Penta of h Hexa	$ABC+(t+p+h)D+(p+h)E+hF$	$t+p+h$ to $n-3t-4p-5h$	$(t+p)$ Tetra of h Penta	$p+h$ Cols.	$(1+p+h)$ hed'id Nos.	the $1+t+p+h$	$\frac{n-5p-6h}{r}$
1	0	0	0	6	Surface Triangles of 0 Hexa	ABC	0 to n $=10$	10th Line Arith. Triang				$\frac{n}{6}=1$ Hexa
2	1	0	0	4	1st Tetra of 0 Hexa	ABC+1D	1 " $n-1\cdot3=n-3=7$	2d Line Surface Triangle	2 Terms	Nat. Trian. Pyr. Etc.	2d	$\frac{10}{4}=2$ Tetra
3	2	0	0		2d Tetra of 0 Hexa	ABC+2D	2 " $n-2\cdot3=n-6=4$	3d Line Surface Triangle	4 Terms	Nat. Trian. Pyr. Etc.	3d	
4	0	1	0	5	1 Penta of 0 Hexa	ABC+1D+1E	1 " $n-1\cdot4=n-4=6$	1st Tetra	1 Col.	Natural	2d	$\frac{10}{5}=2$ Penta
5	1	1	0	4	1st Tetra of 1st Penta of 0 Hexa	ABC+2D+1E	2 " $n-1\cdot3-1\cdot4=n-7$ $=3$	2d Tetra	1 Col.	Natural	3d	$\frac{10-1\cdot5}{4}=1$ Tetra.
6	0	2	0		2d Penta of 0 Hexa	ABC+2D+2E	2 " $n-2\cdot4=n-8=2$	2d Tetra	2 Cols.	Triangular	3d	
7	0	0	1		Surface Triangles of 1st Hexa	ABC+1D+1E+1F	1 " $n-1\cdot5=n-5=5$	1st Penta	1 Col.	Natural	2d	
8	1	0	1	4	1st Tetra of 1st Hexa	ABC+2D+1E+1F	2 " $n-1\cdot3-1\cdot5=n-8$ $=2$	1st Tetra of 1st Penta	1 Col.	Natural	3d	$\frac{10-1\cdot6}{4}=1$ Tetra

TABLE XV. Preliminary Scheme, Hexahedroid for $n=10$, $k=6$.
Derived from General Formulas. (Five Dimensions).

Each Hexa is bounded by 6 Corners, 15 Edges, 20 Surfaces, 15 Tetrahedra, 6 Pentahedroids.
" Penta " " " 5 " 10 " 10 " 5 "
" Tetra " " " 4 " 6 " 4 "

TABLE XV (Continued).

Hexahedroid for $n = 10$, $k = 6$.

Diag. 1. Surface Triangles of Surface or Zero Hexa Shell.

Typical Triangle ABC, same as Diag. 1, Table VIII, but

6 Corners	containing	1 cell	each	= 6	Cells.
15 Edges	"	9 cells	"	= 135	"
20 Surfaces	"	36 cells	"	= 720	"
Total of the 20 Surface Triangles				= 861	"

Each Corner	has	$1 \times 6 =$	6	Combinations.
" Edge	"	$1,022 \times 15 =$	15,330	"
" Surface	"	$55,980 \times 20 =$	1,119,600	"
Total of Surface Triangles		=	1,134,936	"

The Triangles are lettered,

ABC	ACD	ADE	AEF	BCD	BDE	BEF	CDE	CEF	DEF
ABD	ACE	ADF		BCE	BDF		CDF		
ABE	ACF			BCF					
ABF									

Diag. 2. First Inner Tetra Shell of Surface Hexa.

Typical Triangle ABC + 1D, same as Diag. 2, Table VIII, but the Shell is composed of 15 such Pyramids, contains hence

$15 \times 74 = 1110$ cells, $15 \times 591,720 = 8,875,800$ Combinations.

The Pyramids are lettered,

ABCD	ABDE	ABEF	ACDE	ACEF	ADEF	
ABCE	ABDF		ACDF			
ABCF						
			BCDE	BCEF	BDEF	CDEF
			BCDF			

Each Pyramid is composed of 4 Triangles, hence 120 in all.

Diag. 3. Second Inner Tetra of Surface Hexa.

Typical Triangle ABC + 2D, same as Diag. 3, Table IX, but 15 times repeated, gives:

$15 \times 10 = 150$ Cells, $15 \times 226,800 = 3,402,000$ Combinations

Lettering similar to First Tetra.

Sum of the two Tetra Shells = 1260 Cells, with, 12,277,800 Comb.

Diag. 4. Surface Triangles of First Inner Penta Shell of Surface Hexa.

Typical Triangle ABC + 1D + 1E, same as Diag. 4, Table XII, but Shell is composed of 6 such Pentahedroids, hence contains

$6 \times 5 = 30$ Corner Cells
$6 \times 40 = 240$ Edge Cells
$6 \times 60 = 360$ Surface Cells
$6 \times 105 = 630$ Total Cells.

TABLE XV (Continued).

$6 \times$ 25,200 = 151,200 Combinations, in Corners
$6 \times$ 806,400 = 4,838,400 Combinations, in Edges
$6 \times$ 2,646,000 = 15,876,000 Combinations, in Surfaces
$6 \times$ 3,477,600 = 20,865,600 Combinations in All.

The 6 Pentas having 10 Triangles each give 60 in all.
The 6 Pentas are lettered,

ABCDE ABCDF ABCEF ABDEF ACDEF BCDEF

The 10 Triangles of the first Penta ABCDE are lettered the same as the 10 Triangles of Diag. 4, Table XII. The remaining Pentas are similarly lettered.

Diag. 5. First Inner Tetra of First Penta Shell.
Typical Triangle ABC + 2D + 1E, same as Diag. 5, Table XII.
The 5 Tetra of this Diag. are repeated 6 times, giving:
$6 \times 20 = 120$ Cells, $6 \times 1{,}512{,}000 = 9{,}072{,}000$ Combinations.
The 6 Pentas having 5 Tetras having 4 Surfaces each give $6 \times 5 \times 4 = 120$ Triangles in all.
The 5 Tetras of the first Penta, ABCDE, will be lettered as in Table XII, the others similarly.

Diag. 6. Surface Triangles of Second Inner Penta Shell.
Typical Triangle ABC + 2D + 2E, same as Diag. 6, Table XII.
This Shell 6 times repeated gives:
6×1 Cell = 6 Cells, $6 \times 113{,}400 = 680{,}400$ Combinations.

Sum of the Penta Shells (Diags. 4, 5, 6), give
756 Cells containing 30,618,000 Combinations.

A	1	2	3	4	5	B
1	30240	75600	100800	75600	30240	
2	75600	151200	151200	75600		
3	100800	151200	100800			
4	75600	75600				
5	30240					
C						

TABLE XV

DIAG. 7

Surface Triangles of First Inner Hexa Shell. Typical Triangle ABC+1D+1E+1F.

6 Corners of 1 Cell each = 6 Cells.
15 Edges " 3 Cells each = 45 "
20 Surfaces " 3 Cells each = 60 "
Total = 111 "

6 Corners have 30,240 Comb. each = 181,440 Combinations.
15 Edges " 252,000 " " = 3,780,000 "
20 Surfaces " 453,600 " " = 9,072,000 "
Total = 13,033,440 "

Lettering same as surface triangles of surface hexa, but one example of each of the three missing letters added.

TABLE XV (Concluded).

DIAG. 8.

A	2	B
2	226800	
C		

First Inner Tetra of First Inner Hexa. Typical Triangle ABC+2D+1E+1F.

One Cell only but 15 Tetra, giving Total of 15 Cells. 15×226,800=3,402,000 Total Combinations.

The 15 tetras are lettered same as in Diag. 2. In the single cell composing each tetra are contained two each of the four letters designating the tetra, and one of each of the two missing letters.

Total Combinations of First Inner Hexa Shell
= Sum of Diags. 7 and 8 = 16,435,440.

Summary of Hexahedroid.

6	Classes	of	1	Variation	having	6	Combinations
135	"	"	2	Variations	"	15,330	"
720	"	"	3	"	"	1,119,600	"
1260	"	"	4	"	"	12,277,800	"
756	"	"	5	"	"	30,618,000	"
126	"	"	6	"	"	16,435,440	"
3003	"	"	all	"	"	60,466,176	"
= 11th Hexahedroidal Number.							$=6^{10}$.

By putting h equal to zero in this rule it reduces to the one already given for the pentahedroid.

In Table XV is worked out from the general formulas a hexahedroid for $n=10$. First is given a preliminary table showing the number and kind of diagrams needed. The first line of this table repeats the general formulas from which the whole is derived. No really new diagrams are required until we reach the first inner hexa, and only the surface triangles and the first tetra shell of this, the latter containing too only 1 cell, are developed. It might perhaps be more interesting to use a higher value of n so as to develop more of the inner shells, but the numbers increase so rapidly in size that space forbids. For example, if we used $n=15$ the total of all the combinations would be $6^{15}=470{,}184{,}984{,}576$ and 16 diagrams would be required.

Let $k=7$ or higher. We might go on giving k successively higher values, and so develop a septa, an octa, a nona, etc. But the methods would always be the same, and in every case we should end with a general rule that included all of the previous ones. Hence we may at once give the perfectly general rule that will include all the preceding and all the succeeding, viz.:

The typical triangle of the t tetra, of the p penta, of the h

hexa, of the s septa, of the o octa,.......of the $q(k-1)$-hedroidal shell, of the $f(k)$-hedroidal shell.

(1) *will be lettered* $ABC+(t+p+h+s+o+........+f)D + (p+h+s+o+.......+f)E + (h+s+o+......+f)F + (s+o+......+f)G + ...+f$ times the kth letter.

(2) *will be numbered* from $t+p+h+s+o+........+f$ to $n-3t-4p-5h-6s-7o-........-(k-1)f$.

(3) *will be derived from* the typical triangle of the $(t+p)$ tetra of the h penta, of the s hexa, of the o septa,...... ......of the $q(k-1)$-hedroidal shell, by discarding the first $p+h+s+o+......+f$ columns and multiplying the remaining columns successively, by the $(1+p+h+s+o+ ....+f)$-hedroidal numbers in order, beginning with the $(1+t+p+h+s+o+......+f)$th.

(4) *and will have on each edge,* $n-4t-5p-6h-7s-8o........-kf+1$ cells.

(5) The number of r-hedroidal shells required will be $\frac{n-5p-6h-7s-8o......-fk}{r}$, where for r is to be substituted the order of the shell required, and the corresponding letter of the shell in the numerator is then to be omitted.

To apply these rules simply give to the letters t, p, h, s, etc., successively the values, 0, 1, 2, 3, etc., in all combinations, until negative values occur, or until the proper number of shells have been developed. As far as lettering and numbering are concerned, these rules apply to all cases. For derivation they apply only to the inner shells after the surface tetra. The latter and the surface triangles must be calculated line by line, according to the general rule given on page 21. By considering the surface triangles and tetra to be made up of triangular shells, and considering a typical *edge* of such shells, calling the outer edge the zero shell, a perfectly general rule could be given for all cases. But it would be cumbrous, so that practically we find it better to divide the derivation, as has been done, into two rules.

One may well question whether all the foregoing is very important or useful. Certainly it is not of very great advantage until high values of n and k are reached. Still even in fairly simple cases it is of some help. To show this, Table XVI has been given for $n=4$, $k=6$. This shows all the ways in which four dice may be

thrown. Here we reach only the *first* surface tetra, and even this has only one cell, viz., the case where all four of the dice show a different number. The number of ways in which this can occur is given directly by $n! = 24$. All the other classes are shown in the surface triangles. There are 6 where one number only appear, 45 where two appear, 60 where three different numbers appear, and 15 where all four dice show different numbers. All the calculations can be made mentally, for when in the surface triangle we have said $6 \times 2 = 12$, we have obtained all of the different numbers. The method of representation enables all the classes to be enumerated without difficulty or doubt, and gives all the detailed information that can be desired. The total number of classes is 126, or the fifth hexahedroidal number. The total of all combinations is $1296 = 6^4$.

Table XVI.

Hexahedroid for $n=4$, $k=6$.

I	0	1	2	3	4	II
0	1	4	6	4	1	
1	4	12	12	4		
2	6	12	6			
3	4	4				
4	1					
III						

Diag. 1. Surface Triangles.
Typical Triangle I, II, III.

6 Corners	$\times$ 1 =	6 Comb.	6×1 =	6	Classes
15 Edges	$\times$ 14 =	210 "	15×3 =	45	"
20 Surfaces	$\times$ 36 =	720 "	20×3 =	60	"
Total	=	936 "	=	111	"

I	1	II
1	24	
III		

Diag. 2. First Surface Tetra Shell.
Typical Triangle I, II, III, + 1 IV.

15 Tetras $\times$ 24	=	360 Comb.	15×1 =	15	Classes
Total of All	=	1296 "	=	126	"
	=	6^4	=	5th Hexahedroidal No.	

In fact, it was the inquiry of a friend with regard to dice that started the whole investigation.

But whether one concedes to this system any measure of usefulness or not, it affords a striking example of the wonderful interrelations between numbers and geometry, and adds another to the many remarkable properties of the arithmetical triangle. Pascal, in the work referred to at the beginning, exclaimed: *"C'est une chose étrange combien il est fertile en propriétés! Chacun peut s'y exercer."* We have exercised ourselves there, and hope that this new property or extension may not prove wholly uninteresting.

M. Mott-Smith.

George Washington University, Washington, D. C.